8.1 - Preliminary Theory–Linear Systems

Due Sat

Parallels exist between concepts, theorems, and vocabulary of linear differential equations and linear systems of differential equations:

- Existence and uniqueness of solutions
- The superposition principle
- Linear dependence/independence and how they relate to the Wronskian (though the Wronskian is different here)
- Fundamental set of solutions
- Homogeneous and nonhomogeneous systems
- Complementary and particular solutions

Recall from algebra: A system of equations can be represented as a matrix equation. For instance, the system

$$\begin{cases} 2x + 3y - 5z = 12 \\ 7x + 4y + 6z = 18 \\ -2x \qquad\;\; + \;\; z = -27 \end{cases}$$ can be represented by the matrix equation

$$\begin{pmatrix} 2 & 3 & -5 \\ 7 & 4 & 6 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 18 \\ -27 \end{pmatrix} \qquad \begin{pmatrix} 2x + 3y - 5z \\ 7x + 4y + 6z \\ -2x \qquad + z \end{pmatrix} = \begin{pmatrix} 12 \\ 18 \\ -27 \end{pmatrix}$$

and more generally as $A\vec{X} = \vec{B}$

Constant matrix (vector)

Coefficient matrix

Variable matrix (vector)

In this chapter, we will primarily be concerned with linear systems of DEs that have the forms

$\mathbf{X}' = \mathbf{A}\mathbf{X}$ and $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$

nonhomogeneous

$\vec{X}' = A\vec{X}$

vector notation

Homogeneous

$\vec{X}' = A\vec{X} + \vec{F}$

Matrix

vectors

Example: Write the given linear system in matrix form.

$$\begin{cases} \dfrac{dx}{dt} = 4x - 7y \\ \dfrac{dy}{dt} = 5x \end{cases}$$

$\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\vec{X}' = \begin{pmatrix} 4 & -7 \\ 5 & 0 \end{pmatrix} \vec{X}$

Example: Write the given linear system in matrix form.

$$\begin{cases} \dfrac{dx}{dt} = -3x + 4y + e^{-t}\sin 2t \\ \dfrac{dy}{dt} = 5x + 9z + 4e^{-t}\cos 2t \\ \dfrac{dz}{dt} = y + 6z - e^{-t} \end{cases}$$

nonhomogeneous

Form: $\vec{X}' = \boxed{A}\,\vec{X} + \vec{F}(t)$

$\vec{X}' = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix} \vec{X} + \begin{pmatrix} \sin 2t \\ 4\cos 2t \\ -1 \end{pmatrix} e^{-t}$

OR $\begin{pmatrix} e^{-t}\sin 2t \\ 4e^{-t}\cos 2t \\ -e^{-t} \end{pmatrix}$

Example: Write the given linear system without the use of matrices.

$$\mathbf{X}' = \begin{pmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{5t} - \begin{pmatrix} 8 \\ 0 \\ 3 \end{pmatrix} e^{-2t}$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= 7x + 5y - 9z - 8e^{-2t} \\ \frac{dy}{dt} &= 4x + y + z + 2e^{5t} \\ \frac{dz}{dt} &= -2y + 3z + e^{5t} - 3e^{-2t} \end{aligned}}$$

Example: Verify that the vector **X** is a solution of the given homogeneous linear system.

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{X}; \qquad \mathbf{X} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^t = \begin{pmatrix} e^t + 4te^t \\ 3e^t - 4te^t \end{pmatrix}$$

LHS: $\vec{X}' = \begin{pmatrix} e^t + 4e^t + 4te^t \\ 3e^t - 4e^t - 4te^t \end{pmatrix} = \begin{pmatrix} 5e^t + 4te^t \\ -e^t - 4te^t \end{pmatrix}$

RHS: $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} e^t + 4te^t \\ 3e^t - 4te^t \end{pmatrix} = \begin{pmatrix} 2e^t + 8te^t + 3e^t - 4te^t \\ -e^t - 4te^t \end{pmatrix}$

$$= \begin{pmatrix} 5e^t + 4te^t \\ -e^t - 4te^t \end{pmatrix}$$

LHS = RHS ✓